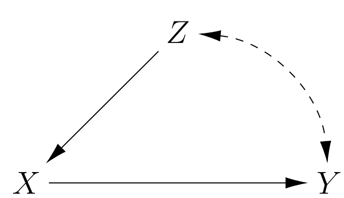
Causal Inference for Health Data (Winter 2026)
STATS C160/C260
Homework 2 Solutions

Exercise 1. SCMs and the Truncated Product

Consider the structural causal model below[1], where $a, b \in (0,1)$ are parameters.

$$
\begin{aligned}
\mathbf{V} &= \{Z, X, Y\} \\
\mathbf{U} &= \{U_x, U_y, U_z, U_{zy}\} \\
\mathcal{F} &= \begin{cases} f_Z = U_z \oplus U_{zy} \\ f_X = U_x \oplus Z \\ f_Y = (U_y \wedge (X \oplus U_{zy})) \vee (\neg U_y \wedge (X \oplus \neg U_{zy})) \end{cases} \\
P(\mathbf{u}) &= \{P(U_z = 1) = a, P(U_x = 1) = b, P(U_y = 1) = 1/5, P(U_{zy} = 1) = 3/4\}
\end{aligned}
$$

(a) Draw the corresponding causal diagram.



(b) Write the following queries as functions of a and b:

There are several ways in which this question can be approached, the following is just one of them.

(a) $P(Y = 1)$

Without any intervention:

$$f_X = U_x \oplus U_z \oplus U_{zy}, \tag{1}$$
$$f_Y = (U_y \wedge (U_x \oplus U_z)) \vee (\neg U_y \wedge \neg(U_x \oplus U_z)), \tag{2}$$

[1] $\neg, \wedge, \vee, \oplus$ represents the logical operators negation, and, or, xor respectively.

then

$$
\begin{aligned}
& P(Y=1) && (3)\\
&= \sum_{u_x,u_z,u_y} P(Y=1 \mid u_x,u_z,u_y)P(u_x,u_z,u_y) && (4)\\
&= \sum_{u_x,u_z,u_y} P((u_y \wedge (u_x \oplus u_z)) \vee (\neg u_y \wedge \neg(u_x \oplus u_z)) = 1 \mid u_x,u_z,u_y)P(u_x,u_z,u_y) && (5)\\
&= \sum_{u_x,u_z} \left(P(u_x \oplus u_z = 1 \mid u_x,u_z)P(U_y=1) + P(\neg(u_x \oplus u_z) = 1 \mid u_x,u_z)P(U_y=0)\right) P(u_x,u_z) && (6)\\
&= (P(U_x=0,U_z=1) + P(U_x=1,U_z=0))P(U_y=1) && (7)\\
&\quad + (P(U_x=0,U_z=0) + P(U_x=1,U_z=1))P(U_y=0) && (8)\\
&= \tfrac{1}{5}((1-b)a + b(1-a)) + \tfrac{4}{5}(ab + (1-a)(1-b)) && (9)\\
&= \tfrac{1}{5}(a+b-2ab) + \tfrac{4}{5}(1-a-b+2ab) && (10)\\
&= \tfrac{6}{5}ab - \tfrac{3}{5}(a+b) + \tfrac{4}{5} && (11)
\end{aligned}
$$

(b) $P(Y=1 \mid X=1)$

If $X=1$

$$f_Y = (U_y \wedge \neg U_{zy}) \vee (\neg U_y \wedge U_{zy}) \quad (12)$$

$$
\begin{aligned}
& P(Y=1,X=1) && (13)\\
&= \sum_{u_x,u_z,u_y,u_{zy}} P(Y=1 \mid X=1,u_y,u_{zy})P(X=1 \mid u_x,u_z,u_{zy})P(u_x,u_z,u_y,u_{zy}) && (14)\\
&= \sum_{u_x,u_z,u_y,u_{zy}} P((u_y \wedge \neg u_{zy}) \vee (\neg u_y \wedge u_{zy}) = 1 \mid u_y,u_{zy})\\
&\qquad P(u_x \oplus u_z \oplus u_{zy} = 1 \mid u_x,u_z,u_{zy})P(u_x,u_z,u_y,u_{zy}) && (15)\\
&= \sum_{u_{zy}} \sum_{u_y} P((u_y \wedge \neg u_{zy}) \vee (\neg u_y \wedge u_{zy}) = 1 \mid u_y,u_{zy})P(u_y)\\
&\qquad \sum_{u_x,u_z} P(u_x \oplus u_z \oplus u_{zy} = 1 \mid u_x,u_z,u_{zy})P(u_x,u_z)P(u_{zy}) && (16)\\
&= P(U_{zy}=0)P(U_y=1)\left(\sum_{u_x,u_z} P(u_x \oplus u_z = 1 \mid u_x,u_z)P(u_x,u_z)\right) +\\
&\quad P(U_{zy}=1)P(U_y=0)\left(\sum_{u_x,u_z} P(u_x \oplus u_z = 0 \mid u_x,u_z)P(u_x,u_z)\right) && (17)\\
&= P(U_{zy}=0)P(U_y=1)(P(U_z=0)P(U_x=1) + P(U_z=1)P(U_x=0)) +\\
&\quad P(U_{zy}=1)P(U_y=0)(P(U_z=0)P(U_x=0) + P(U_z=1)P(U_x=1)) && (18)\\
&= \left(\tfrac{1}{4}\right)\left(\tfrac{1}{5}\right)((1-a)b + a(1-b)) + \left(\tfrac{3}{4}\right)\left(\tfrac{4}{5}\right)((1-a)(1-b) + ab) && (19)\\
&= \tfrac{1}{20}(b+a-2ab) + \tfrac{3}{5}(2ab-a-b+1) && (20)\\
&= \tfrac{11}{10}ab - \tfrac{11}{20}(a+b) + \tfrac{3}{5} && (21)
\end{aligned}
$$

If $X = 1$ it means that $1 = U_x \oplus U_z \oplus U_{zy}$, hence $U_{zy} = \neg(U_x \oplus U_z)$. Then,

$$
\begin{aligned}
P(X=1) &= \sum_{u_x,u_z,u_{zy}} P(u_x \oplus u_z \oplus u_{zy} = 1) P(u_x, u_z, u_{zy}) && (22)\\
&= P(U_{zy}=0)\left(\sum_{u_x,u_z} P(u_x \oplus u_z = 1 \mid u_x, u_z) P(u_x, u_z)\right) + \\
&\quad P(U_{zy}=1)\left(\sum_{u_x,u_z} P(u_x \oplus u_z = 0 \mid u_x, u_z) P(u_x, u_z)\right) && (23)\\
&= P(U_{zy}=0)(P(U_z=0)P(U_x=1) + P(U_z=1)P(U_x=0)) + \\
&\quad P(U_{zy}=1)(P(U_z=0)P(U_x=0) + P(U_z=1)P(U_x=1)) && (24)\\
&= \left(\tfrac{1}{4}\right)((1-a)b + a(1-b)) + \left(\tfrac{3}{4}\right)((1-a)(1-b) + ab) && (25)\\
&= \tfrac{1}{4}(b + a - 2ab) + \tfrac{3}{4}(2ab - a - b + 1) && (26)\\
&= ab - \tfrac{1}{2}(a+b) + \tfrac{3}{4}. && (27)
\end{aligned}
$$

To conclude, we have

$$
P(Y=1 \mid X=1) = \frac{P(Y=1, X=1)}{P(X=1)} = \frac{\frac{11}{10}ab - \frac{11}{20}(a+b) + \frac{3}{5}}{ab - \frac{1}{2}(a+b) + \frac{3}{4}} \qquad (28)
$$

(c) $P(Y = 1 \mid X = 1, Z = 1)$

Let's use another approach to solve this one. Let's write $P(\mathbf{u}, \mathbf{v})$ in terms of a and b:

U_z	U_x	U_y	U_{zy}	Z	X	Y	P
0	0	0	0	0	0	1	$(4/5)(1/4)(1-a)(1-b)$
0	0	0	1	1	1	1	$(4/5)(3/4)(1-a)(1-b)$
0	0	1	0	0	0	0	$(1/5)(1/4)(1-a)(1-b)$
0	0	1	1	1	1	0	$(1/5)(3/4)(1-a)(1-b)$
0	1	0	0	0	1	0	$(4/5)(1/4)(1-a)b$
0	1	0	1	1	0	0	$(4/5)(3/4)(1-a)b$
0	1	1	0	0	1	1	$(1/5)(1/4)(1-a)b$
0	1	1	1	1	0	1	$(1/5)(3/4)(1-a)b$
1	0	0	0	1	1	0	$(4/5)(1/4)a(1-b)$
1	0	0	1	0	0	0	$(4/5)(3/4)a(1-b)$
1	0	1	0	1	1	1	$(1/5)(1/4)a(1-b)$
1	0	1	1	0	0	1	$(1/5)(3/4)a(1-b)$
1	1	0	0	1	0	1	$(4/5)(1/4)ab$
1	1	0	1	0	1	1	$(4/5)(3/4)ab$
1	1	1	0	1	0	0	$(1/5)(1/4)ab$
1	1	1	1	0	1	0	$(1/5)(3/4)ab$

Then:

$$P(Y = 1 \mid X = 1, Z = 1) \tag{29}$$
$$= \frac{P(Y = 1, X = 1, Z = 1)}{P(X = 1, Z = 1)} \tag{30}$$
$$= \frac{(4/5)(3/4)(1-a)(1-b) + (1/5)(1/4)a(1-b)}{(4/5)(3/4)(1-a)(1-b) + (1/5)(3/4)(1-a)(1-b) + (4/5)(1/4)a(1-b) + (1/5)(1/4)a(1-b)} \tag{31}$$
$$= \frac{[(3/5)(1-a) + (1/20)a](1-b)}{[(3/4)(1-a) + (1/4)a](1-b)} = \frac{(3/5)(1-a) + (1/20)a}{(3/4)(1-a) + (1/4)a}. \tag{32}$$

(d) $P(Y = 1 \mid do(X = 1)) - P(Y = 1 \mid do(X = 0))$

For this part we need to consider the model $\mathcal{M}_x$

$$\mathbf{V} = \{Z, X, Y\}$$
$$\mathbf{U} = \{U_x, U_y, U_z, U_{zy}\}$$
$$\mathcal{F} = \begin{cases} f_Z = U_z \oplus U_{zy} \\ f_X = x \\ f_Y = (U_y \wedge (X \oplus U_{zy})) \vee (\neg U_y \wedge (X \oplus \neg U_{zy})) \end{cases}$$
$$P(\mathbf{u}) = \{P(U_z = 1) = a, P(U_x = 1) = b, P(U_y = 1) = 1/5, P(U_{zy} = 1) = 3/4\}$$

There

$$f_Y = (U_y \wedge (x \oplus U_{zy})) \vee (\neg U_y \wedge (x \oplus \neg U_{zy})), \tag{33}$$

consequently:

$$P(Y_{X=1} = 1) = \sum_{u_y, u_{zy}} P(Y_{X=1} = 1 \mid u_y, u_{zy}) P(u_y, u_{zy}) \tag{34}$$
$$= \sum_{u_y, u_{zy}} P((u_y \wedge \neg u_{zy}) \vee (\neg u_y \wedge u_{zy}) = 1 \mid u_y, u_{zy}) P(u_y, u_{zy}) \tag{35}$$
$$= P(U_y = 0, U_{zy} = 1) + P(U_y = 1, U_{zy} = 0) \tag{36}$$
$$= \tfrac{13}{20}, \tag{37}$$
$$P(Y_{X=0} = 1) = \sum_{u_y, u_{zy}} P(Y_{X=0} = 1 \mid u_y, u_{zy}) P(u_y, u_{zy}) \tag{38}$$
$$= \sum_{u_y, u_{zy}} P((u_y \wedge u_{zy}) \vee (\neg u_y \wedge \neg u_{zy}) = 1 \mid u_y, u_{zy}) P(u_y, u_{zy}) \tag{39}$$
$$= P(U_y = 0, U_{zy} = 0) + P(U_y = 1, U_{zy} = 1) \tag{40}$$
$$= \tfrac{7}{20}; \tag{41}$$

hence

$$P(Y_{X=1} = 1) - P(Y_{X=0} = 1) = \tfrac{3}{10}. \tag{42}$$

(c) If any of the probability distributions in the previous question are independent of a or b, explain if that independence can be inferred from the causal diagram alone (not knowing the actual function, just their observable and unobservable arguments). Your answer should address each query.

(a) $P(Y = 1)$ is not independent of a and b. This is related to the fact that Y is not independent of U_x and U_z due to the paths $U_x \to X \to Y$ and $U_z \to Z \to X \to Y$ in the graph (if we explicitly consider the $\mathbf{U}$').

(b) $P(Y = 1 \mid X = 1)$ is not independent of a and b. We can see this in the graph from the fact that the paths $U_x \to X \leftarrow Z \leftarrow U_{zy} \to Y$ and $U_z \to Z \leftarrow U_{zy} \to Y$ are d-connected given Z.

(c) For $P(Y = 1 \mid X = 1, Z = 1)$, the probability is independent of b which is associated with U_x. We can see this in the graph (augmented with the $\mathbf{U}$s) because $(Y \perp\!\!\!\perp U_x \mid X, Z)$.

(d) $P(Y_{X=1} = 1) - P(Y_{X=0} = 1)$ is independent of a and b that are associated with U_z and U_x, respectively. This can be seen from the graph (showing the $\mathbf{U}$s explicitly) since $(Y \perp\!\!\!\perp U_z, U_x)$ in $G_{\overline{X}}$.

Exercise 2. Applying the Backdoor Criterion

Consider the following definition:

Definition 1 (Minimal adjustment set)**.** A minimal adjustment set $\mathbf{Z}$ relative to a pair of variables X and Y is a set of variables that satisfies the back-door criterion to find the causal effect of X on Y, such that no proper subset of $\mathbf{Z}$ satisfies the criterion.

Consider SCMs compatible with the graphs in Fig. 1. Find a *minimal* adjustment set to compute the causal effect of X on Y and write the corresponding expression for $P(y \mid do(x))$.

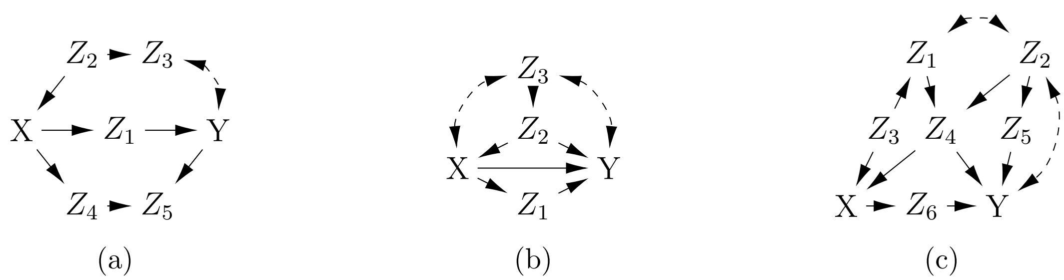


Figure 1: Graphs for the SCM used in this problem.

(a) $\mathbf{Z} = \emptyset$ is a minimal backdoor admissible set.

$$P(y \mid do(x)) = P(y \mid x)$$

(b) There is not set $\mathbf{Z}$ that satisfies the backdoor criterion in this model.

(c) A minimal set is $\mathbf{Z} = \{Z_3, Z_4\}$.

$$P(y \mid do(x)) = \sum_{Z_3, Z_4} P(y \mid x, z_3, z_4) P(z_3, z_4)$$

Exercise 3. Back-door Adjustment – I

(a) Construct a causal diagram $\mathcal{G}$ such that a set $\mathbf{Z}$ does not satisfy the Back-door criterion relative to (X, Y), but the adjustment formula with the same $\mathbf{Z}$ still holds

$$P(y \mid do(x)) = \sum_z P(y \mid x, z) P(z)$$

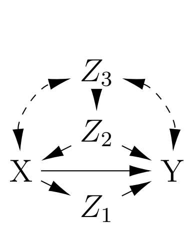
for every SCM $\mathcal{M}$ with causal diagram $\mathcal{G}$). For the constructed example, prove that that the adjustment formula holds using do-calculus.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ Z & \end{tikzcd}$$

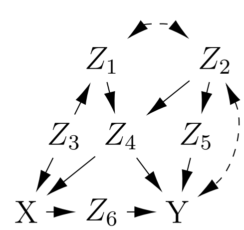
$\mathbf{Z} = \{Z\}$ does not satisfy the back-door criterion, but

$$\begin{aligned}
P(y \mid do(x)) &= P(y \mid x) && \text{Rule 2: } (Y \perp\!\!\!\perp X)_{\mathcal{G}_{\underline{X}}} \quad (43)\\
&= P(y \mid x) \sum_z P(z) && \text{Multiply by 1} \quad (44)\\
&= \sum_z P(y \mid x) P(z) && \text{Move term into the sum} \quad (45)\\
&= \sum_z P(y \mid x, z) P(z) && (Y \perp\!\!\!\perp Z \mid X) \quad (46)
\end{aligned}$$

(b) Consider the following causal diagrams. For each case find a *minimal* back-door admissible (if one exists) set to compute the causal effect of X on Y, and write the corresponding expression for $P(y \mid do(x))$.



(i)



(ii)

(i)

There is no set that satisfies the backdoor criterion in this model.

(ii)

A minimal set is $\mathbf{Z} = \{Z_3, Z_4\}$.

$$P(y \mid do(x)) = \sum_{Z_3, Z_4} P(y \mid x, z_3, z_4) P(z_3, z_4)$$

(c) Prove the Back-door criterion using do-calculus.

Suppose $\mathbf{Z}$ satisfies the Back-door criterion relative to $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{G}$.

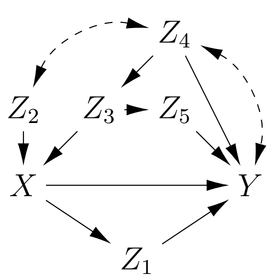
$$\begin{aligned}
P(\mathbf{y} \mid do(\mathbf{x})) &= \sum_{\mathbf{z}} P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}) P(\mathbf{z} \mid do(\mathbf{x})) && \text{Condition on } \mathbf{Z} && (47)\\
&= \sum_{\mathbf{z}} P(\mathbf{y} \mid do(\mathbf{x}), \mathbf{z}) P(\mathbf{z}) && \text{Rule 3: } (\mathbf{Z} \perp\!\!\!\perp \mathbf{X})_{\mathcal{G}_{\overline{\mathbf{X}}}} && (48)\\
&= \sum_{\mathbf{z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z}) && \text{Rule 2: } (\mathbf{Y} \perp\!\!\!\perp \mathbf{X} \mid \mathbf{Z})_{\mathcal{G}_{\underline{\mathbf{X}}}} && (49)
\end{aligned}$$

Conditioning holds for any probability distribution. Rule 3 is licensed by the separation statement $(\mathbf{Z} \perp\!\!\!\perp \mathbf{X})$ in $\mathcal{G}_{\overline{\mathbf{X}}}$. By condition (a) of the BD criterion, we know that no variable in $\mathbf{Z}$ is a descendant of $\mathbf{X}$ and since all paths with arrows into $\mathbf{X}$ are disconnected in $\mathcal{G}_{\overline{\mathbf{X}}}$ any path between $X \in \mathbf{X}$ and $Z \in \mathbf{Z}$ starts with an arrow going out of X and find a collider before reaching Z, otherwise Z would be a descendant of $\mathbf{X}$. But such collider must be inactive because nothing is being conditioned on. We conclude there is no path contradicting the required separation.

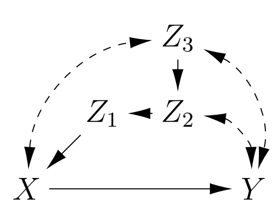
For Rule 2 we need $(\mathbf{Y} \perp\!\!\!\perp \mathbf{X} \mid \mathbf{Z})_{\mathcal{G}_{\underline{\mathbf{X}}}}$ which is precisely condition (b), since in $\mathcal{G}_{\underline{\mathbf{X}}}$ only the backdoor paths remain.

Exercise 4. Back-door Adjustment – II

(a) Consider the following causal diagrams. For each case find a *minimal* back-door admissible (if one exists) set to compute the causal effect of X on Y, and write the corresponding expression for $P(y \mid do(x))$.



(i)



(ii)

(i)

A minimal set is $\mathbf{Z} = \{Z_2, Z_3\}$.

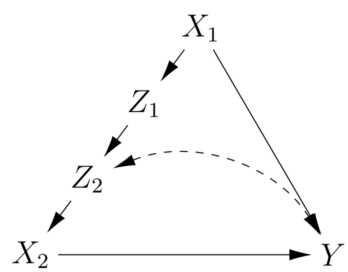
$$P(y \mid do(x)) = \sum_{Z_2, Z_2} P(y \mid x, z_2, z_3) P(z_2, z_3)$$

(ii)

There is not set $\mathbf{Z}$ that satisfies the backdoor criterion in this model.

(b) Consider the causal diagram $\mathcal{G}$ below. The goal is to identify the effect of $\mathbf{X} = \{X_1, X_2\}$ on Y. First, determine whether there exists a set $\mathbf{Z}$ that satisfies the back-door criterion. Second, using the rules of do-calculus, show that

$$P(y \mid do(x_1, x_2)) = \sum_{z_1, z_2} P(y \mid x_1, x_2, z_1, z_2) P(z_1, z_2).$$



There is no set $\mathbf{Z}$ that satisfies the back-door criterion. $\mathbf{Z} = \{Z_1, Z_2\}$ satisfies the adjustment criterion.

$$
\begin{aligned}
&P(y \mid do(x_1, x_2)) && (50)\\
&= P(y \mid do(x_1, x_2, z_1)) && \text{R3: } (Y \perp\!\!\!\perp Z_1 \mid X_1, X_2)_{G_{\overline{X_1 X_2 Z_1}}} \quad (51)\\
&= \sum_{z_2} P(y \mid z_2, do(x_1, x_2, z_1)) P(z_2 \mid do(x_1, x_2, z_1)) && \text{marginalize } Z_2 \quad (52)\\
&= \sum_{z_2} P(y \mid x_1, x_2, z_1, z_2) P(z_2 \mid do(x_1, x_2, z_1)) && \text{R2: } (Y \perp\!\!\!\perp X_1, X_2, Z_1 \mid Z_2)_{G_{\overline{Z_2}\underline{X_1 X_2 Z_1}}} \quad (53)\\
&= \sum_{z_2} P(y \mid x_1, x_2, z_1, z_2) P(z_2 \mid do(z_1)) && \text{R3: } (Z_2 \perp\!\!\!\perp X_1, X_2 \mid Z_1)_{G_{\overline{Z_1 X_1 X_2}}} \quad (54)\\
&= \sum_{z_2} P(y \mid x_1, x_2, z_1, z_2) P(z_2 \mid z_1) && \text{R2: } (Z_2 \perp\!\!\!\perp Z_1)_{G_{\underline{Z_1}}} \quad (55)\\
&= \sum_{z_1, z_2} P(y \mid x_1, x_2, z_1, z_2) P(z_2 \mid z_1) P(z_1) && \text{see note below} \quad (56)\\
&= \sum_{z_1, z_2} P(y \mid x_1, x_2, z_1, z_2) P(z_1, z_2) && (57)
\end{aligned}
$$

Note: Line 56 holds because since $P(y \mid do(x_1, x_2)) = \sum_{z_2} P(y \mid x_1, x_2, z_1, z_2)P(z_2 \mid z_1)$ for arbitrary z_1, we can obtain the same result by taking a weighted average over all values of z_1. In this case, we use the weight $P(z_1)$.

Exercise 5. Do-Calculus Rules

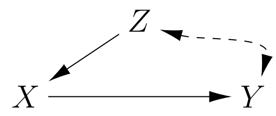
(a) For each of the rules of Do-Calculus, provide an example of a (invalid) use of the rule where the corresponding, licensing separation in the graph does not hold.

Specifically, provide a fully specified SCM and compute the left and right hand side of the equation in the rule in each case, showing that they are different for some particular value of the variables.

You are allowed to construct three different SCMs or re-use them throughout the rules.

Consider the following SCM and causal diagram

$$
\begin{aligned}
\mathbf{V} &= \{Z, X, Y\}\\
\mathbf{U} &= \{U_x, U_y, U_z, U_{zy}\}\\
\mathcal{F} &= \begin{cases} f_Z = U_z \oplus U_{zy} \\ f_X = U_x \oplus Z \\ f_Y = (U_y \wedge (X \oplus U_{zy})) \vee (\neg U_y \wedge (X \oplus \neg U_{zy})) \end{cases}\\
P(\mathbf{u}) &= \{P(U_z = 1) = a, P(U_x = 1) = b, P(U_y = 1) = 1/5, P(U_{zy} = 1) = 3/4\}
\end{aligned}
$$



Rule 1
$P(Y=1 \mid do(X=1), Z=1) = P(Y=1 \mid do(X=1))$ would be allowed by rule 1 if the separation $(Y \perp\!\!\!\perp Z \mid X)_{G_{\overline{X}}}$, which does not hold (due to $Z \dashleftarrow\dashrightarrow Y$. Let's check:

$$
\begin{aligned}
&P(Y=1 \mid do(X=1), Z=1)\\
=&\frac{(4/5)(3/4)(1-a)+(1/5)(1/4)a}{(4/5)(3/4)(1-a)+(1/5)(3/4)(1-a)+(4/5)(1/4)a+(1/5)(1/4)a}\\
=&\frac{11a-12}{10a-15},\\
&P(Y=1 \mid do(X=1))\\
=&(4/5)(3/4)(1-a)+(1/5)(1/4)(1-a)+(4/5)(3/4)a+(1/5)(1/4)a\\
=&\tfrac{13}{20}.
\end{aligned}
$$

Which are different for most a.

Rule 2
$P(Y=1 \mid do(X=1)) = P(Y=1 \mid X=1)$ would be licensed by rule 2 if the separation $(Y \perp\!\!\!\perp X)_{G_{\underline{X}}}$, which does not hold (consider $X \leftarrow Z \dashleftarrow\dashrightarrow Y$). We check the quantities (from the previous question and problem 1(b)(2)):

$$
\begin{aligned}
P(Y=1 \mid do(X=1)) &= \tfrac{13}{20},\\
P(Y=1 \mid X=1) &= \frac{\frac{11}{10}ab - \frac{11}{20}(a+b)+\frac{3}{5}}{ab-\frac{1}{2}(a+b)+\frac{3}{4}},
\end{aligned}
$$

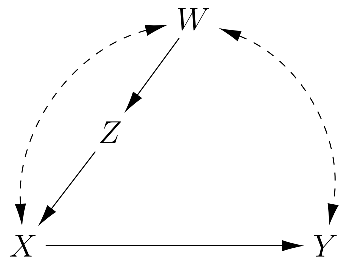
that are different for most values of a and b.

Rule 3
$P(Y=1 \mid do(X=1)) = P(Y=1)$ would be given by rule 3 if $(Y \perp\!\!\!\perp X)_{G_{\overline{X}}}$, which does not hold here. Again, from previous questions we have:

$$
\begin{aligned}
P(Y=1 \mid do(X=1)) &= \tfrac{13}{20},\\
P(Y=1) &= \tfrac{6}{5}ab - \tfrac{3}{5}(a+b)+\tfrac{4}{5};
\end{aligned}
$$

that are different for most a and b.

(b) Consider the following causal diagram:



(i) Explain why neither Z, W, nor $\{Z, W\}$ are back-door admissible sets for the causal effect of X on Y.

Note that if we condition on W, the path $X \leftarrow\dashrightarrow W \leftarrow\dashrightarrow Y$ is open, so sets W and $\{Z, W\}$ are ruled out. If we condition on Z, which is descendant of W, we again open the path $X \leftarrow\dashrightarrow W \leftarrow\dashrightarrow Y$.

(ii) Derive the identification expression for $P(y \mid do(x))$ using the rules of do-calculus. *Hint: in the first step, justify adding $do(w), do(z)$ to $P(y \mid do(x))$. Then, note that $do(x)$ in the $do(w), do(z)$ is the same as $see(x)$. Finally, use the law of conditional probability to compute the numerator and the denominator.*

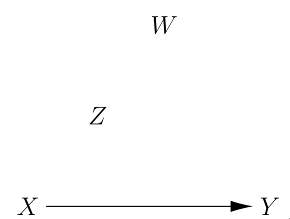
We first want to apply Rule 3 of do-calculus to show that:

$$P(y \mid do(x)) = P(y \mid do(x), do(z), do(w)).$$

Intuitively, this equality will hold if Z, W have *no causal paths* to Y in the $do(x)$ world. Formally, to verify this claim, we need to check whether the independence

$$(Y \perp\!\!\!\perp Z, W \mid X)_{G_{\overline{XZW}}}$$

holds. The graph $G_{\overline{XZW}}$ is given by



Further, we next want to use Rule 2 to show that

$$P(y \mid do(x), do(z), do(w)) = P(y \mid x, do(z), do(w)),$$

Intuitively, this should be true since there are *only causal paths* from X to Y in the $do(z, w)$ world. Formally, to verify this claim, we need to check whether

$$(Y \perp\!\!\!\perp X \mid Z, W)_{G_{\overline{ZW}\underline{X}}}$$

holds. The graph $G_{\overline{ZW}\underline{X}}$,

$$\begin{tikzcd} & W & \\ & Z \arrow[dl] & \\ X & & Y \end{tikzcd}.$$

so the independence holds. Finally,

$$P(y \mid x, do(z), do(w)) = \frac{P(y, x \mid do(z), do(w))}{P(x \mid do(z), do(w))}.$$

Now, $P(y, x \mid do(z), do(w)) = P(y, x \mid do(z))$ by Rule 3, since in the $do(z)$ world, W has *no causal paths* to X, Y. Formally, we check that

$$(Y, X \perp\!\!\!\perp W \mid Z)_{G_{\overline{ZW}}}.$$

The graph $G_{\overline{ZW}}$ is given by

$$\begin{tikzcd} & W & \\ & Z \arrow[dl] & \\ X \arrow[rr] & & Y \end{tikzcd}.$$

which is true. For computing $P(y, x \mid do(z))$, we note that W is valid back-door set for the causal effect of $Z \to \{X, Y\}$. In terms of do-calculus, we expand on W and apply Rule 2, meaning that

$$\begin{aligned} P(y, x \mid do(z), do(w)) &= \sum_w P(y, x \mid do(z), w) P(w \mid do(z)) \\ &= \sum_w P(y, x \mid do(z), w) P(w) \quad \text{Rule 3: } (W \perp\!\!\!\perp Z)_{G_{\overline{Z}}} \\ &= \sum_w P(y, x \mid z, w) P(w) \quad \text{Rule 2: } (Y, X \perp\!\!\!\perp Z \mid W)_{G_{\underline{Z}}} \end{aligned}$$

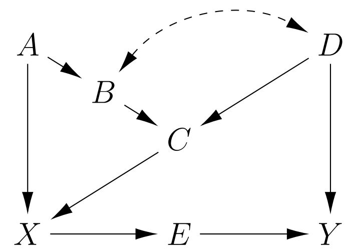
The effect $P(x \mid do(z), do(w))$ can be computed by just summing over y in the above, and equals $P(x \mid do(z), do(w)) = \sum_w P(x \mid z, w) P(w)$. Therefore, putting things together, we obtain

$$P(y \mid do(x)) = \frac{\sum_w P(y, x \mid z, w) P(w)}{\sum_w P(x \mid z, w) P(w)},$$

with the expression being true for any fixed value of $Z = z$.

Exercise 6. Changing the Granularity of the Model

Consider the causal diagram $\mathcal{G}$ below.



(a) Determine whether the causal effect $P(y \mid do(x))$ is identifiable; if so, show how.

There are several choices here, for instance

$$P(y \mid do(x)) = \sum_{a,c} P(y \mid x, a, c) P(a, c), \tag{58}$$

due to the back-door criterion and the fact that $\{A, C\}$ admissible. Or simply because this is the adjustment by the parents. We could also have

$$P(y \mid do(x)) = \sum_{e} P(e \mid x) \sum_{x'} P(y \mid e, x') P(x'), \tag{59}$$

since E is front-door admissible (See Primer page 66). We could also derive some expressions with do-calculus.

(b) Write a SCM $\mathcal{M}$ that induces this causal diagram and a probability distribution $P(\mathbf{V})$ such that for every $\mathbf{v}, P(\mathbf{v}) > 0$. You don't need to show $P(\mathbf{v})$ in your answer.

For instance, consider the following model $\mathcal{M}$:

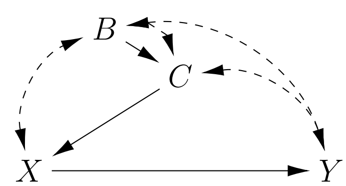
$$\begin{aligned}
\mathbf{V} &= \{X, Y, A, B, C, D, E\} \\
\mathbf{U} &= \{U_b, U_{bd}, U_x, U_y, U_a, U_c, U_e\} \\
\mathcal{F} &= \begin{cases} f_A = U_a \\ f_B = A \oplus U_{bd} \oplus U_b \\ f_C = B \oplus D \oplus U_c \\ f_D = U_{bd} \\ f_X = A \oplus C \oplus U_x \\ f_E = X \oplus U_e \\ f_Y = E \oplus D \oplus U_y \end{cases} \\
P(\mathbf{u}) &= \{P(U_b = 1) = P(U_{bd} = 1) = P(U_x = 1) = P(U_y = 1) \\
&= P(U_a = 1) = P(U_c = 1) = P(U_e = 1) = 1/2\}
\end{aligned} \tag{60}$$

Suppose that in a different study the same system (represented by the SCM $\mathcal{M}$) is observed, but only the variables $\mathbf{V}' = \{X, Y, B, C\}$ are measured.

(c) Write a new SCM $\mathcal{M}' = \langle \mathbf{V}', \mathbf{U}', \mathcal{F}', P(\mathbf{u}') \rangle$ corresponding to the model for this study, based on your answer to the previous question.

$$\begin{aligned}
\mathbf{V}' &= \{X, Y, B, C\} \\
\mathbf{U}' &= \{A, D, U_x, U_y, U_c, U_e\} \\
\mathcal{F} &= \begin{cases} f_B = A \oplus D \\ f_C = B \oplus D \oplus U_c \\ f_X = U_x \oplus A \oplus C \\ f_Y = (X \oplus U_e) \oplus D \oplus U_y \end{cases} \\
P(\mathbf{u}) &= \{P(A=1) = P(D=1) = P(U_x=1) = P(U_y=1) \\
&= P(U_c=1) = P(U_e=1) = 1/2\}
\end{aligned} \tag{61}$$

(d) Draw the causal diagram $\mathcal{G}'$ corresponding to $\mathcal{M}'$.



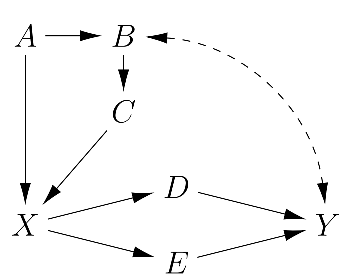
Naturally, since we are talking about the same underlying reality (the SCM you proposed), $\mathcal{M}'$ and $\mathbf{V}'$ have to preserve all the (probabilistic) independencies among the variables in $\mathbf{V}'$ that were implied in $\mathcal{M}$ and $\mathbf{V}$.

(e) Is the effect $P(y \mid do(x))$ identifiable from $P(\mathbf{v}')$ and $\mathcal{G}'$? Specifically, is there a back-door or front-door adjustment set?

The effect $P(y \mid do(x))$ is not identifiable from the graph induced by $\mathcal{M}'$ and $P(\mathbf{V}')$, so there is no admissible sets for back-door adjustment, front-door adjustment, any do-calculus derivation and Identify fails with this instance.

Exercise 7. Many Paths Lead to ID

The target causal quantity $P(y \mid do(x))$ may not be identifiable from $P(\mathbf{V})$ depending on the causal diagram. On the other hand, for identifiable effects, there could be more than one expression that is equal to the effect of interest. Consider the following causal diagram:

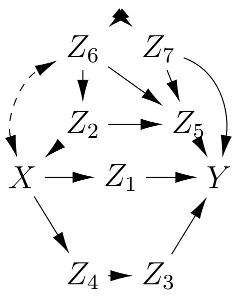


Give 5 different functions of the observational distribution $P(\mathbf{v})$ that are equal to the effect Q. That is, find 5 different identfication expressions. Justify the validity of each expression using the Back-door Criterion, the Front-door criterion, or showing a derivation in do-calculus.

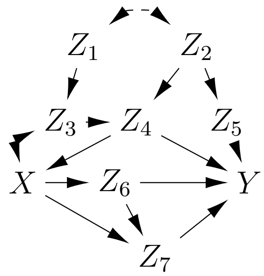
> There are three back-door admissible sets, namely, $\{A, C\}, \{A, B\}, \{A, B, C\}$ that yield
>
> $$P(y \mid do(x)) = \sum_{a,c} P(y \mid x, a, c)P(a, c) \tag{62}$$
>
> $$P(y \mid do(x)) = \sum_{a,b} P(y \mid x, a, b)P(a, b) \tag{63}$$
>
> $$P(y \mid do(x)) = \sum_{a,b,c} P(y \mid x, a, b, c)P(a, b, c) \tag{64}$$
>
> Also, the set $\{D, E\}$ is front-door admissible, and yields:
>
> $$P(y \mid do(x)) = \sum_{d,e} P(d, e \mid x) \sum_{x'} P(y \mid x', d, e)P(x') \tag{65}$$
>
> More entertaining, is the following derivation:
>
> $$\begin{aligned} P(y \mid do(x)) &= P(y \mid do(x), do(c)) && \text{R3: } (Y \perp\!\!\!\perp C \mid X)_{G_{\overline{X,C}}} && (66) \\ &= P(y \mid x, do(c)) && \text{R2: } (Y \perp\!\!\!\perp X \mid C)_{G_{\overline{C}\underline{X}}} && (67) \\ &= \frac{P(y, x \mid do(c))}{P(x \mid do(c))} && \text{Cond. Prob.} && (68) \\ &= \frac{\sum_b P(y, x \mid do(c), b)P(b \mid do(c))}{\sum_b P(x \mid do(c), b)P(b \mid do(c))} && \text{Conditioning on } B && (69) \\ &= \frac{\sum_b P(y, x \mid c, b)P(b \mid do(c))}{\sum_b P(x \mid c, b)P(b \mid do(c))} && \text{R2: } (X, Y \perp\!\!\!\perp C \mid B)_{G_{\underline{C}}} && (70) \\ &= \frac{\sum_b P(y, x \mid c, b)P(b)}{\sum_b P(x \mid c, b)P(b)} && \text{R3: } (B \perp\!\!\!\perp C)_{G_{\overline{C}}} && (71) \end{aligned}$$

Exercise 8. [Optional] Optimal Experimental Design

An advertisement company aims to predict the effect of a new campaign X on the click through rate Y. They have two hypotheses about how the strategy relates to a possibly measured set of covariates $\mathbf{Z}$. The hypotheses are represented in the causal diagrams below (a,b):



(a)



(b)

Variable	**Cost**
X	2
Y	1
Z_1	3
Z_2	1
Z_3	3
Z_4	1
Z_5	6
Z_6	2
Z_7	2

(c)

(a) If it exists, find a minimal admissible set for adjustment in each of the graphs.

For model (a) $\mathbf{Z} = \{Z_5, Z_7\}$ is a minimal back-door adjustment set. For model (b) $\mathbf{Z} = \{Z_4, Z_5\}$ is minimal back-door adjustment set.

(b) The company wants to minimize the measurement cost for identifying $P(y \mid do(x))$. Find the minimum cost ID expression based on the table (c) and justify your selection.

For model (a) the cheapest is to use the front-door adjustment with $\mathbf{Z} = \{Z_1, Z_4\}$ that has cost 4 with the expression:

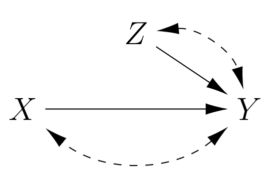
$$P(y \mid do(x)) = \sum_{z_1, z_4} P(z_1, z_4 \mid x) \sum_{x'} P(y \mid x', z_1, z_4) P(x')$$

For model (b) the cheapest is to perform back-door adjustment with $\mathbf{Z} = \{Z_2, Z_4\}$ that has cost 2, and derives the expression:

$$P(y \mid do(x)) = \sum_{z_2, z_4} P(y \mid x, z_2, z_4) P(z_2, z_4)$$

Exercise 9. [Optional] Non-Identifiability

(a) Prove that the effect $P(y \mid do(x))$ is not identifiable from the following causal diagram and $P(X, Y, Z)$, where the domain of X, Y, Z is $\{0, 1\}$ and $P(x, y, z) > 0, \forall x, y, z \in \{0, 1\}$.



Define an SCM $\mathcal{M}_1$ as follows:

$$M_1 = \begin{cases} U_X, U_Y, U_Z \in \{0,1\} \\ Z \leftarrow U_Z \\ X \leftarrow U_X \\ Y \leftarrow X \oplus Z \oplus U_X \oplus U_Z \oplus U_Y \\ P(U_X = 1) = P(U_Z = 1) = 0.5, P(U_Y = 1) = 0.9 \end{cases} \tag{72}$$

where $\oplus$ is the "*xor*" operator. In this SCM $\mathcal{M}_1$, fix U_X, U_Y, U_Z, the evaluation of $Y(\mathbf{U})$ is given by

$$Y \leftarrow X \oplus Z \oplus U_X \oplus U_Z \oplus U_Y = U_X \oplus U_Z \oplus U_X \oplus U_Z \oplus U_Y = U_Y. \tag{73}$$

This implies that for $i, j, k \in \{0,1\}$,

$$P_{M_1}(X = i, Y = j, Z = k) = P(U_X = i)P(U_Y = j)P(U_Z = k). \tag{74}$$

Fix an intervention $do(X = x)$, the evaluation of $Y_x(\mathbf{U})$ is given by

$$Y \leftarrow x \oplus Z \oplus U_X \oplus U_Z \oplus U_Y = x \oplus U_Z \oplus U_X \oplus U_Z \oplus U_Y = x \oplus U_X \oplus U_Y. \tag{75}$$

This implies that for $i, j \in \{0,1\}$,

$$P_{M_1}(Y = j \mid do(X = i)) = 0.5. \tag{76}$$

Now define an SCM $\mathcal{M}_2$ as follows:

$$M_1 = \begin{cases} U_X, U_Y, U_Z \in \{0,1\} \\ Z \leftarrow U_Z \\ X \leftarrow U_X \\ Y \leftarrow U_Y \\ P(U_X = 1) = P(U_Z = 1) = 0.5, P(U_Y = 1) = 0.9 \end{cases} \tag{77}$$

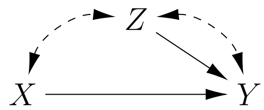
Evidently, $\mathcal{M}_1, M_2$ coincide in $P(x, y, z)$. That is, for $i, j, k \in \{0,1\}$,

$$P_{M_1}(X = i, Y = j, Z = k) = P_{M_2}(X = i, Y = j, Z = k) = P(U_X = i)P(U_Y = j)P(U_Z = k).$$

On the other hand, $\mathcal{M}_1, M_2$ generate different interventional distribution $P(y \mid do(x))$, i.e.,

$$P_{M_1}(Y = 1 \mid do(X = 0)) = 0.5 \tag{78}$$
$$P_{M_2}(Y = 1 \mid do(X = 0)) = P_{M_2}(Y = 1) = 0.9. \tag{79}$$

(b) Prove that the effect $P(y \mid do(x))$ is not identifiable from the following causal diagram and $P(X, Y, Z)$, where the domain of X, Y, Z is $\{0,1\}$ and $P(x, y, z) > 0, \forall x, y, z \in \{0,1\}$.



Define an SCM $\mathcal{M}_1$ as follows:

$$M_1 = \begin{cases} U_X, U_Y, U_Z \in \{0,1\} \\ Z \leftarrow U_X \oplus U_Z \\ X \leftarrow U_X \\ Y \leftarrow X \oplus Z \oplus U_Z \oplus U_Y \\ P(U_X = 1) = P(U_Z = 1) = 0.5, P(U_Y = 1) = 0.9 \end{cases} \tag{80}$$

where $\oplus$ is the "*xor*" operator. In this SCM $\mathcal{M}_1$, fix U_X, U_Y, U_Z, the evaluation of $Y(\mathbf{U})$ is given by

$$Y \leftarrow X \oplus Z \oplus U_Z \oplus U_Y = U_X \oplus U_X \oplus U_Z \oplus U_Z \oplus U_Y = U_Y. \tag{81}$$

This implies that for $i, j, k \in \{0,1\}$,

$$P_{M_1}(X = i, Y = j, Z = k) = P(U_X = i)P(U_Y = j)P(U_Z = k). \tag{82}$$

Fix an intervention $do(X = x)$, the evaluation of $Y_x(\mathbf{U})$ is given by

$$Y \leftarrow x \oplus Z \oplus U_Z \oplus U_Y = x \oplus U_X \oplus U_Z \oplus U_Z \oplus U_Y = x \oplus U_X \oplus U_Y. \tag{83}$$

This implies that for $i, j \in \{0,1\}$,

$$P_{M_1}(Y = j \mid do(X = i)) = 0.5. \tag{84}$$

Now define an SCM $\mathcal{M}_2$ as follows:

$$M_1 = \begin{cases} U_X, U_Y, U_Z \in \{0,1\} \\ Z \leftarrow U_X \oplus U_Z \\ X \leftarrow U_X \\ Y \leftarrow U_Y \\ P(U_X = 1) = P(U_Z = 1) = 0.5, P(U_Y = 1) = 0.9 \end{cases} \tag{85}$$

Evidently, $\mathcal{M}_1, M_2$ coincide in $P(x, y, z)$. That is, for $i, j, k \in \{0,1\}$,

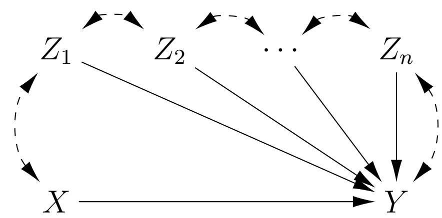
$$P_{M_1}(X = i, Y = j, Z = k) = P_{M_2}(X = i, Y = j, Z = k) = P(U_X = i)P(U_Y = j)P(U_Z = k).$$

On the other hand, $\mathcal{M}_1, M_2$ generate different interventional distribution $P(y \mid do(x))$, i.e.,

$$P_{M_1}(Y = 1 \mid do(X = 0)) = 0.5 \tag{86}$$

$$P_{M_2}(Y = 1 \mid do(X = 0)) = P_{M_2}(Y = 1) = 0.9. \tag{87}$$

(c) Prove that the effect $P(y \mid do(x))$ is not identifiable from the following causal diagram and $P(X, Y, Z_1, \ldots, Z_n)$, $1 \leq n < \infty$, where the domain of $X, Y, Z_1, \ldots, Z_n$ is $\{0, 1\}$ and $P(x, y, z_1, \ldots, z_n) > 0, \forall x, y, z_1, \ldots, z_n \in \{0, 1\}$. *Note: We discussed in class different strategies for constructing such counter-examples. You are allowed to leverage this understanding but need to construct your own pair of models.*



Define an SCM $\mathcal{M}_1$ as follows:

$$M_1 = \begin{cases} U_X, U_Y \in \{0, 1\} \\ U_i \in \{0, 1\}, i = 0, 1, \ldots, n \\ X \leftarrow U_0 \\ Z_i = U_{i-1} \oplus U_i, i = 1, \ldots, n \\ Y \leftarrow (\oplus_{i=1}^n Z_i) \oplus X \oplus U_n \oplus U_Y \\ P(U_i = 1) = 0.5, i = 0, 1, \ldots, n \\ P(U_Y = 1) = 0.9 \end{cases} \tag{88}$$

where $\oplus$ is the "*xor*" operator. In this SCM $\mathcal{M}_1$, fix $U_Y, U_0, U_1, \ldots, U_{n+1}$, the evaluation of $Y(\mathbf{U})$ is given by

$$\begin{aligned} Y &\leftarrow (\oplus_{i=1}^n Z_i) \oplus X \oplus U_Y \oplus U_{n+1} && (89) \\ &= (\oplus_{i=1}^n (U_{i-1} \oplus U_i)) \oplus U_0 \oplus U_n \oplus U_Y && (90) \\ &= (U_0 \oplus U_1) \oplus (U_1 \oplus U_2) \oplus \cdots \oplus (U_{n-1} \oplus U_n) \oplus U_0 \oplus U_n \oplus U_Y && (91) \\ &= U_0 \oplus U_n \oplus U_0 \oplus U_n \oplus U_Y && (92) \\ &= U_Y && (93) \end{aligned}$$

This implies that for $i, j, k_1, k_2, \ldots, k_n \in \{0, 1\}$,

$$\begin{aligned} &P_{M_1}(X = i, Y = j, Z_1 = k_1, \ldots, Z_n = k_n) && (94) \\ &= P_{M_1}(X = i, Z_1 = k_1, \ldots, Z_n = k_n) P(U_Y = j) && (95) \\ &= 0.5^{n+1} P(U_Y = j). && (96) \end{aligned}$$

The last step holds since U_i, $i = 0, 1, \ldots, n$, are binary variable uniformly drawn over $\{0, 1\}$. Fix an intervention $do(X = x)$, the evaluation of $Y_x(\mathbf{U})$ is given by

$$\begin{aligned} Y &\leftarrow (\oplus_{i=1}^n Z_i) \oplus x \oplus U_Y \oplus U_{n+1} && (97) \\ &= (\oplus_{i=1}^n (U_{i-1} \oplus U_i)) \oplus x \oplus U_n \oplus U_Y && (98) \\ &= (U_0 \oplus U_1) \oplus (U_1 \oplus U_2) \oplus \cdots \oplus (U_{n-1} \oplus U_n) \oplus x \oplus U_n \oplus U_Y && (99) \\ &= U_0 \oplus U_n \oplus x \oplus U_n \oplus U_Y && (100) \\ &= x \oplus U_0 \oplus U_Y && (101) \end{aligned}$$

This implies that for $i, j \in \{0,1\}$,

$$P_{M_1}(Y = j \mid do(X = i)) = 0.5. \tag{102}$$

Now define an SCM $\mathcal{M}_2$ as follows:

$$M_1 = \begin{cases} U_X, U_Y \in \{0,1\} \\ U_i \in \{0,1\}, i = 0,1,\ldots,n \\ X \leftarrow U_0 \\ Z_i = U_{i-1} \oplus U_i, i = 1,\ldots,n \\ Y \leftarrow U_Y \\ P(U_i = 1) = 0.5, i = 0,1,\ldots,n \\ P(U_Y = 1) = 0.9 \end{cases} \tag{103}$$

Evidently, $\mathcal{M}_1, M_2$ coincide in $P(x, y, z)$. That is, for $i, j, k_1, k_2, \ldots, k_n \in \{0,1\}$,

$$\begin{aligned} &P_{M_1}(X = i, Y = j, Z_1 = k_1, \ldots, Z_n = k_n) \\ &= P_{M_2}(X = i, Y = j, Z_1 = k_1, \ldots, Z_n = k_n) \\ &= 0.5^{n+1} P(U_Y = j). \end{aligned}$$

On the other hand, $\mathcal{M}_1, M_2$ generate different interventional distribution $P(y \mid do(x))$, i.e.,

$$P_{M_1}(Y = 1 \mid do(X = 0)) = 0.5 \tag{104}$$

$$P_{M_2}(Y = 1 \mid do(X = 0)) = P_{M_2}(Y = 1) = 0.9. \tag{105}$$