Causal Inference for Health Data

(STATS C160/C260 — Winter 2026)

Lecture 11: Causal Effect Heterogeneity

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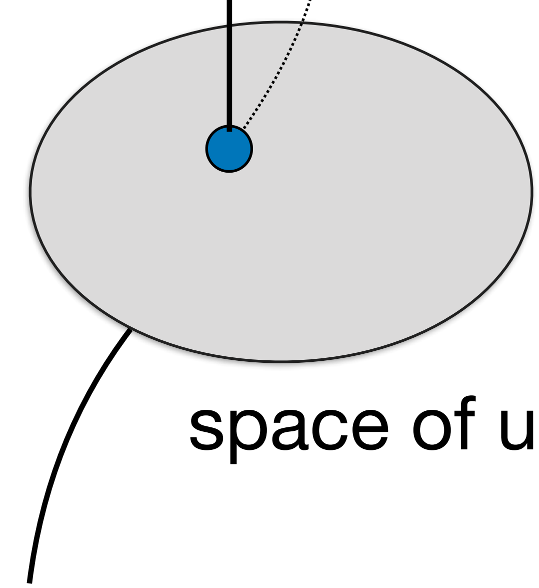
SCM - mechanisms & population

after u is fixed, the evaluation is deterministic

Evaluation of SCM M:

unit $u = (u_1, \ldots, u_k)$

$$V_1 \leftarrow f_1(u_1)$$
$$V_2 \leftarrow f_2(v_1, u_2)$$
$$\vdots$$
$$V_k \leftarrow f_k(v_1, \ldots, v_{k_1}, u_k)$$



space of units $\mathscr{U}$

distribution over units $P(u)$

Mechanisms $\mathscr{F}$ + Distribution $P(u)$ $= M$

Counterfactuals — Structural Basis Expansion

Theorem – Structural Basis Expansion of Counterfactuals.

Consider an SCM M and the counterfactual distribution induced by it.

Let $\mathbf{E}$ be the factual evidence, $\mathbf{C}$ is the counterfactual condition, and $\mathbf{Y}$ the outcome variable. The structural basis expansion for $E[\mathbf{Y}_{\mathbf{C}} \mid \mathbf{E}]$ is written as:

$$E[\mathbf{Y}_{\mathbf{C}} \mid \mathbf{E}] = \sum_{\mathbf{u}} \mathbf{Y}_{\mathbf{C}}(\mathbf{u}) \times P(\mathbf{u} \mid \mathbf{E} = \mathbf{e}).$$

Prediction: evaluating the counterfactual

Action: computing the unit-level counterfactual

Abduction: obtain posterior distribution

Total Effect

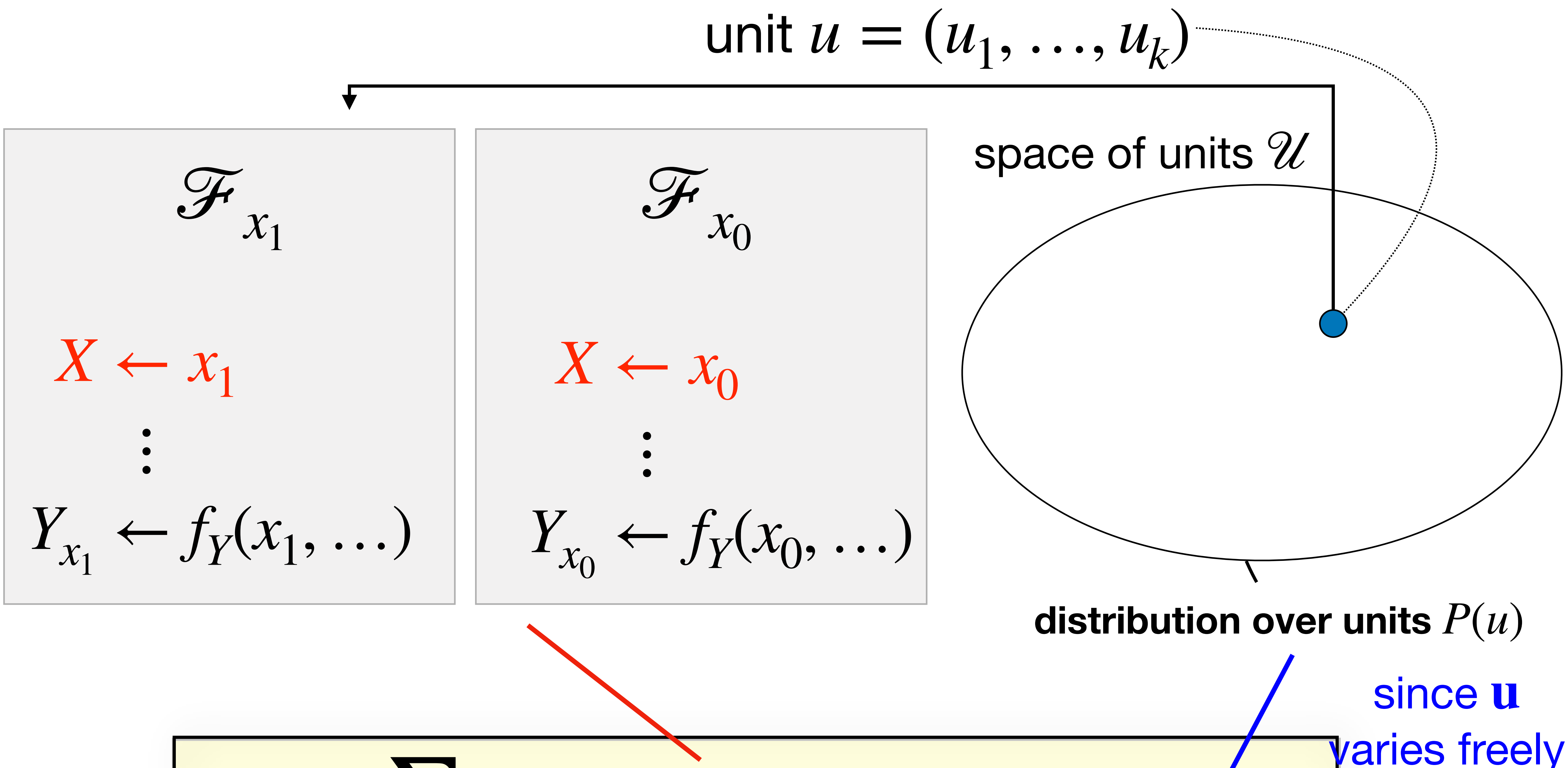
- The **Total Effect** $E(Y \mid do(X = x_1)) - E(Y \mid do(X = x_0))$ is a L_2-quantity representing the causal effect of an $x_0 \rightarrow x_1$ transition on the outcome Y.

- The mechanisms $\mathscr{F}$ are modified by intervention $do(X = x)$, while units $\mathbf{U}$ are drawn randomly from the population.

- The structural basis expansion of TE is:

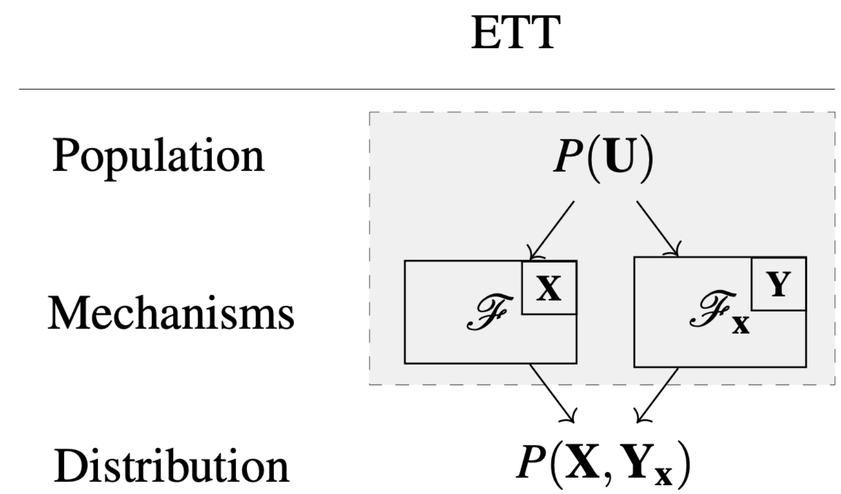
$$TE_{x_0,x_1}(y) = \sum_{\mathbf{u}} [\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})] P(\mathbf{u})$$

Total Effect: Sampling Evaluation



$$\textbf{TE} = \sum_{\mathbf{u}} \left[\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})\right] \times P(\mathbf{u})$$

Effect of the Treatment on the Treated



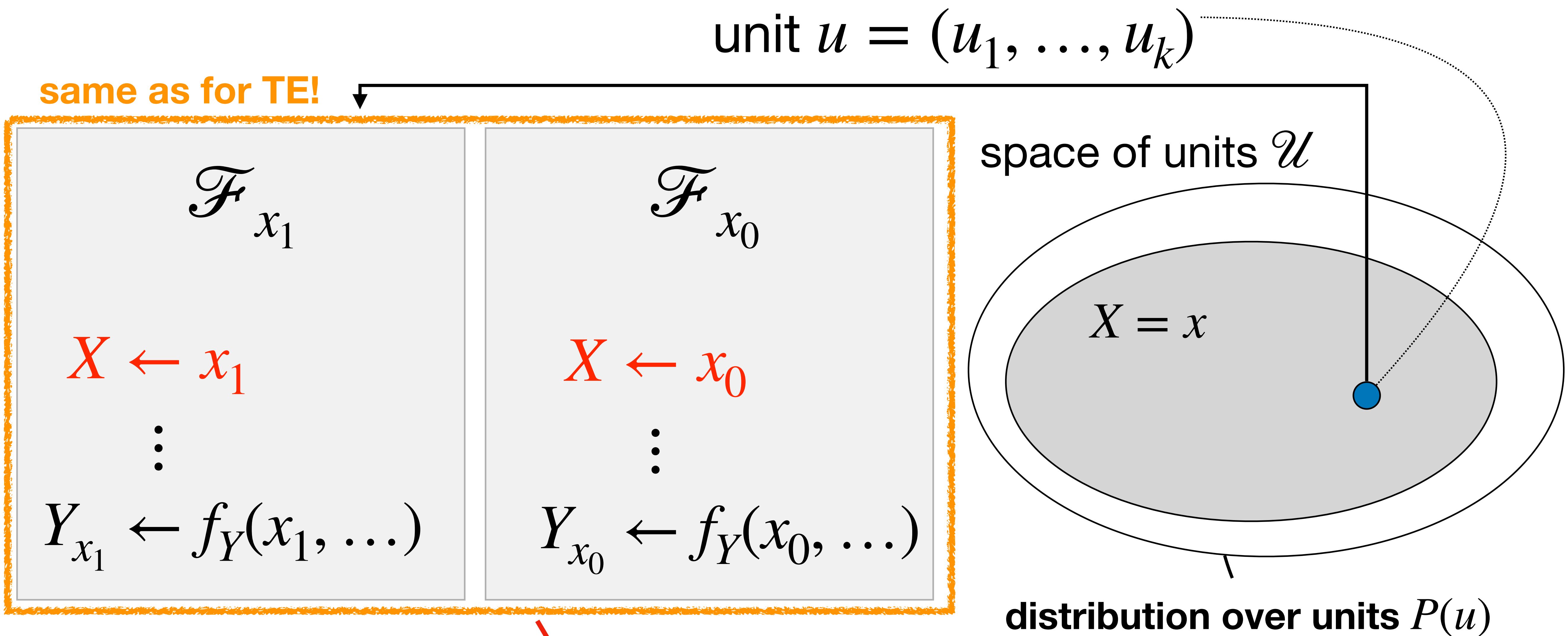
- The ETT mixes the natural and the counterfactual worlds, or L_1 and L_2-variations, and is written as:

$$E(Y_{X=x_1} \mid X = x) - E(Y_{X=x_0} \mid X = x)$$

- The term $E[Y_{X=x} \mid X = x']$ is Layer 3, requiring the joint $P(Y_{X=x}, X = x')$
- Each unit $\mathbf{U} = \mathbf{u}$ passes through the set of mechanisms in the natural and counterfactual worlds, $\mathscr{F}$ and $\mathscr{F}_x$, and its counterfactual expansion is:

$$ETT_{x_0,x_1}(y \mid x') = \sum_{\mathbf{u}} [\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})] P(\mathbf{u} \mid X = x')$$

Effect of Treatment on the Treated: Sampling Evaluation



$$\textbf{ETT} = \sum_{\mathbf{u}} [\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})] \times P(\mathbf{u} \mid X = x)$$

since $\mathbf{u}$ varies with X

Comparing L_2- and L_3-effects (TE vs. ETT)

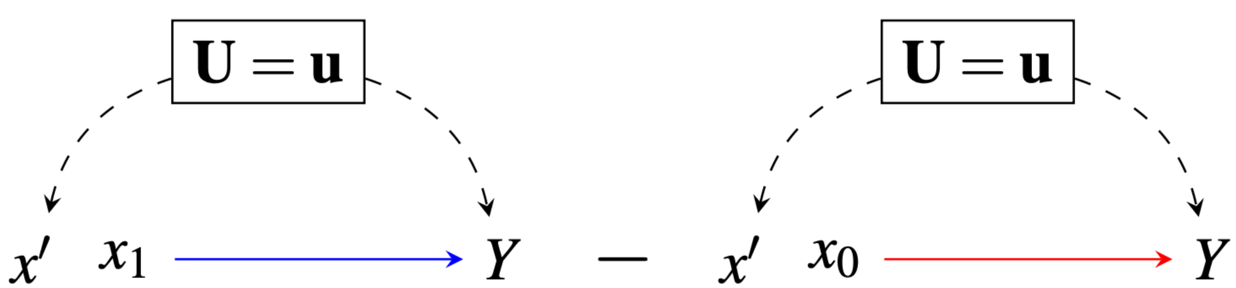
$$TE_{x_0,x_1}(y) = \sum_{\mathbf{u}} [\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})] P(\mathbf{u})$$

$$ETT_{x_0,x_1}(y \mid x) = \sum_{\mathbf{u}} [\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})] P(\mathbf{u} \mid X = x)$$

- The selection of mechanisms (in red) is the same between the ETT and TE ($\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})$) — the downstream effect from X to Y.
- They select different sets of units from the population (in blue),
    - TE_{x_0,x_1} selects units at random from the population ($P(\mathbf{U})$),
    - ETT_{x_0,x_1} selects units that are inclined to naturally decide $X = x$ ($P(\mathbf{U} \mid X = x)$).
- TE and ETT are both measures of causal effects w.r.t. layers L_2 and L_3.

ETT Contrast

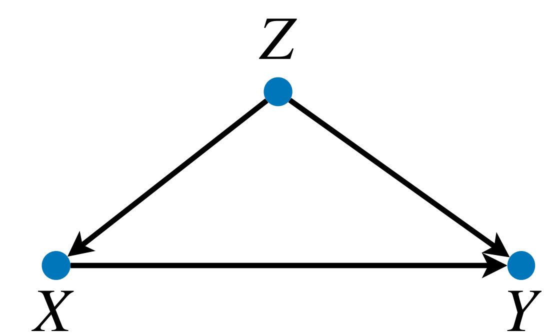
- A common way to use the ETT is to compare the effect of two regimes x_0 and x_1, for a population that would naturally choose a regime x' (possibly equal to x_1 or x_0).
- This contrast can be visualized as:



- On both sides, we select units $\mathbf{U}$ such that $X = x'$, while Y perceives X as being x_1 in the left, and as x_0 in the right.

ETT Identification

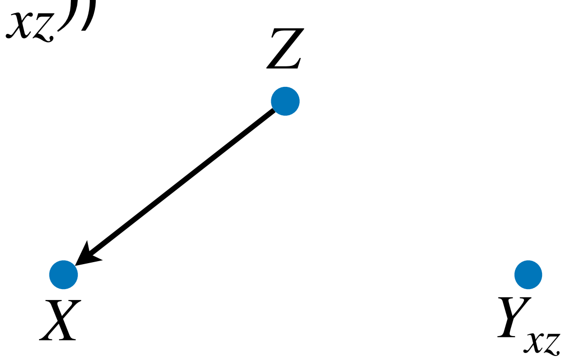
$$P(y_x \mid x') = \sum_z P(y_x \mid z, x')P(z \mid x') \text{ (Conditioning on } Z\text{)}$$



$$= \sum_z P(y_x \mid z_x, x')P(z \mid x') \quad \text{(exclusion: } \{X\} \cap An(Z) = \emptyset\text{)}$$

$$= \sum_z P(y_{xz} \mid z_x, x')P(z \mid x') \quad \text{(consistency } (Z_x = z \Rightarrow Y_x = Y_{xz})\text{)}$$

$$= \sum_z P(y_{xz} \mid z, x')P(z \mid x') \quad \text{(exclusion: } \{X\} \cap An(Z) = \emptyset\text{)}$$



$\mathcal{G}_A(Y_{xz}, X, Z)$

$$= \sum_z P(y_{xz} \mid z, x)P(z \mid x') \quad \text{(indep. } Y_{xz} \perp X \mid Z \text{ in } \mathcal{G}_A\text{)}$$

$$= \boxed{\sum_z P(y \mid z, x)P(z \mid x')} \quad \text{(consistency } (Z = z, X = x \Rightarrow Y_{xz} = Y)\text{)}$$

can be computed from observational data!

z-Total Effect

- The **z-specific Total Effect**

$$E(Y \mid do(X = x_1), Z = z) - E(Y \mid do(X = x_0), Z = z)$$

  is a L_2-quantity representing the distribution of the effect of $do(X = x_1)$ vs. $do(X = x_0)$ on Y for units such that $Z = z$.

- The mechanisms $\mathscr{F}$ are modified by intervention $do(X = x)$, while only units $\mathbf{U}$ s.t. $Z = z$ are drawn from the population:

$$z - TE_{x_0,x_1}(y) = \sum_{\mathbf{u}} [\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})] P(\mathbf{u} \mid Z = z)$$

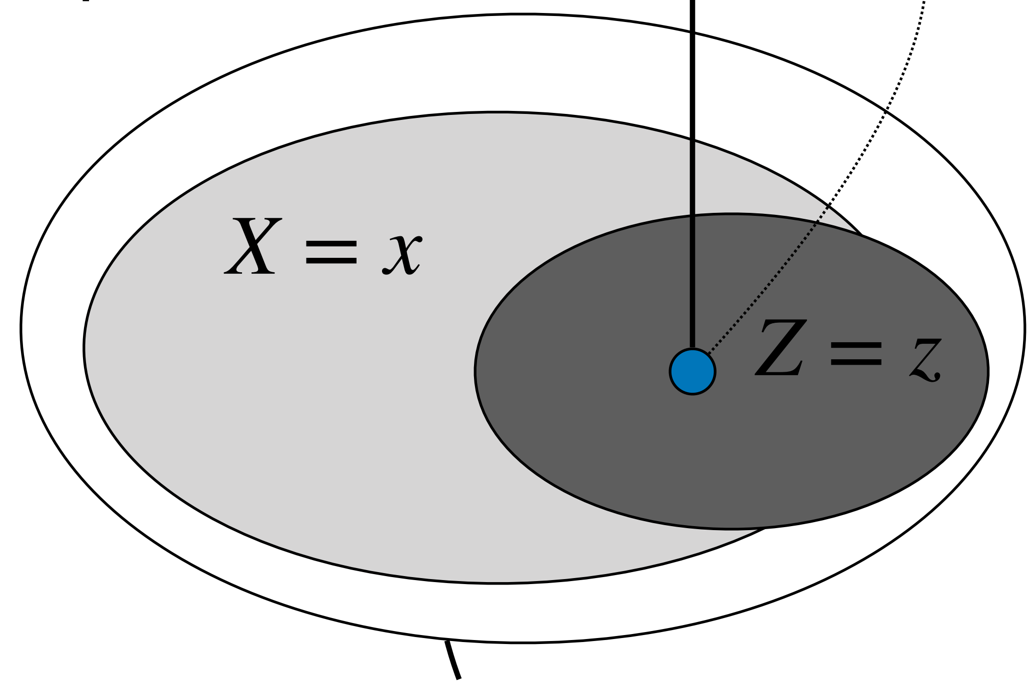
z-Total Effect: Sampling Evaluation

unit $u = (u_1, \ldots, u_k)$

same as for TE/ETT

$\mathscr{F}_{x_1}$	$\mathscr{F}_{x_0}$
$X \leftarrow x_1$	$X \leftarrow x_0$
$\vdots$	$\vdots$
$Y_{x_1} \leftarrow f_Y(x_1, \ldots)$	$Y_{x_0} \leftarrow f_Y(x_0, \ldots)$

space of units $\mathscr{U}$



distribution over units $P(u)$

since $\mathbf{u}$ varies with Z

$$z\text{-}\mathbf{TE} = \sum_{\mathbf{u}} [\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})] \times P(\mathbf{u} \mid Z = z)$$

v-Total Effect

- The **v-specific Total Effect**

$$E(Y_{x_1} \mid V = v) - E(Y_{x_0} \mid V = v)$$

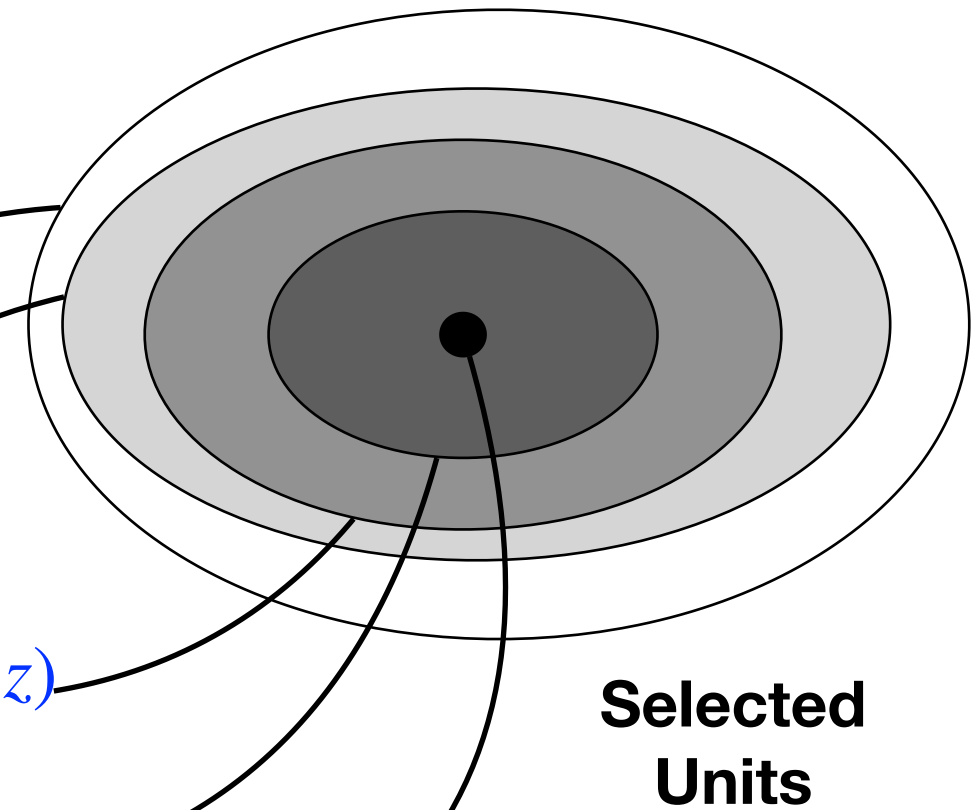
for any $V = v$ is generally an L_3-quantity for the effect of $do(X = x_1)$ vs. $do(X = x_0)$ on Y for units such that $V = v$.

- The mechanisms $\mathscr{F}$ are modified by intervention $do(X = x)$, while only units $\mathbf{U}$ s.t. $V = v$ are drawn from the population:

$$v - TE_{x_0,x_1}(y) = \sum_{\mathbf{u}} [\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})] P(\mathbf{u} \mid V = v)$$

Granularity

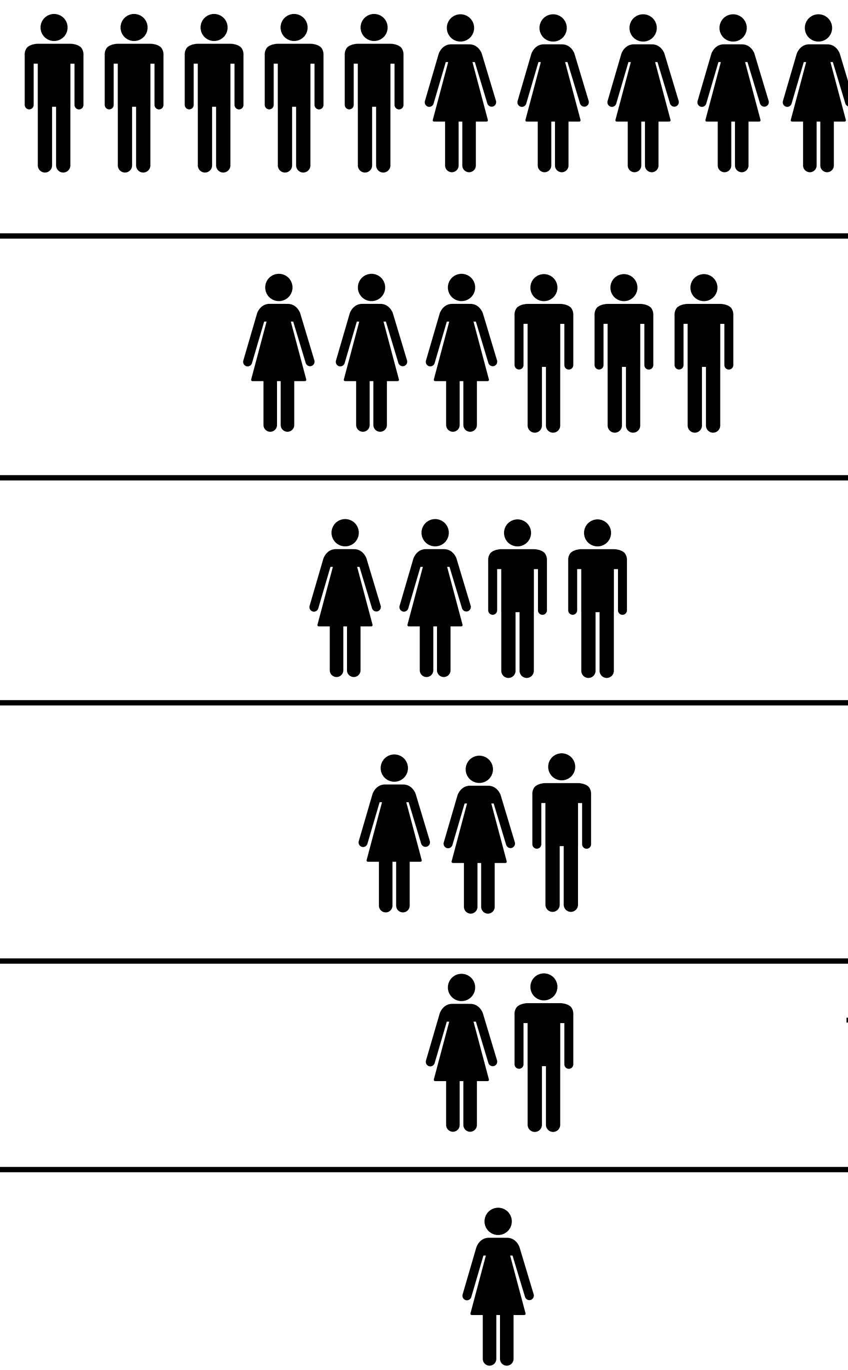
Quantity	Unit-level Difference	Posterior
TE	$\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})$	$P(\mathbf{u})$
ETT	$\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})$	$P(\mathbf{u} \mid X = x)$
x, z-TE	$\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})$	$P(\mathbf{u} \mid X = x, Z = z)$
v-TE	$\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})$	$P(\mathbf{u} \mid V = v)$
u-TE	$\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})$	$\delta_{\mathbf{u}}$



Generic Format:

$$\sum_{\mathbf{u}} [\mathbf{Y}_{X=x_1}(\mathbf{u}) - \mathbf{Y}_{X=x_0}(\mathbf{u})] \times P(\mathbf{u} \mid \mathbf{E} = \mathbf{e})$$

Explainability Plane



Population Axis		
population $P(u)$		entire population
x-subset $P(u \mid x)$		treated population
z, x-subset $P(u \mid x, z)$		treated, below 40 years
z, x, w-subset $P(u \mid x, z, w)$		treated, below 40, with weight loss
v-subset $P(u \mid v)$		treated, below 40, weight loss, HbA1c normal
unit-level u		Catherine, below 40, weight loss, HbA1c normal

Population axis is controlled by the granularity of the event $\mathbf{E} = \mathbf{e}$